Journal of Approximation Theory **113**, 221–228 (2001)
doi:10.1006/jath.2001.3615, available online at http://www.idealibrary.com on IDEAL®

A Sharp Inequality of Markov Type for Polynomials Associated with Laguerre Weight

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Communicated by Tamás Erdélyi

Received July 11, 2000; accepted June 14, 2001

The best possible constant A_n in an inequality of Markov type

$$\left\| \frac{d}{dx}\,(e^{-x} p_n(x)) \right\|_{[0,\,\infty)} \leqslant A_n \,\|e^{-x} p_n(x)\|_{[0,\,\infty)},$$

where $\|\cdot\|_{[0,\,\infty)}$ denotes the sup-norm on the half real line $[0, \infty)$ and p_n is an arbitrary polynomial of degree at most n, is determined in terms of the weighted Chebyshev polynomials associated with the Laguerre weight e^{-x} on $[0, \infty)$.

1. INTRODUCTION

Among the many theorems relating the derivative of a polynomial to the polynomial itself, two of the most well-known inequalities are Bernstein's inequality

$$|p'(x)| \leqslant \frac{n}{\sqrt{1-x^2}}\,\|p\|_{[-1,\,1]}, \qquad -1 < x < 1, \quad p \in \mathscr{P}_n,$$

and Markov's inequality

$$\|p'\|_{[-1,\,1]} \leqslant n^2\,\|p\|_{[-1,\,1]}, \qquad p \in \mathscr{P}^r_n. \tag{1}$$

Here $\mathscr{P}_n$ and $\mathscr{P}^r_n$ denote the polynomials of degree n and the polynomials of degree n with real coefficients, respectively, and $\|\cdot\|_{[-1,\,1]}$ denotes the sup-norm on the interval $[-1, 1]$; that is, $\|p\| = \sup\{|p(x)|: -1 \leqslant x \leqslant 1\}$. Notice that the version of Bernstein's inequality mentioned here is a pointwise

[1] Part of this work is done as a partial fulfillment of the M.S. at the University of Central Florida.

0021-9045/01 $35.00

inequality while Markov's inequality is global. Even though Markov's inequality is global, we find that Markov's inequality gives us a better local estimate than that of Bernstein for x near -1 or 1. But, when x is not near -1 or 1, the estimate given by Bernstein's result can be much better than Markov's. In this paper, we will be concerned with inequalities similar to the latter, the so-called Markov-type inequalities.

In general, the Markov-type inequalities involve finding

$$A_n := \sup \frac{\|(wp)'\|}{\|wp\|} \tag{2}$$

as a function of n, where the supremum is taken over all p from a subset of either $\mathscr{P}_n$ or $\mathscr{P}^r_n$, the weight w is fixed, and $\|\cdot\|$ is a norm. The classical Markov inequality (1) is the case when the supremum is taken over $\mathscr{P}^r_n$, $w \equiv 1$, $\|\cdot\| = \|\cdot\|_{[-1,1]}$, and A_n is determined as

$$A_n = \frac{\|T'_n\|}{\|T_n\|},$$

where $T_n(x) = \frac{1}{2^{n-1}} \cos(n \operatorname{arc} \cos x)$, the nth Chebyshev polynomial of the first kind.

Various problems arise from considering different norms, weights, and sets over which the supremum is taken. When sup-norm is used, there are only two other cases besides the classical Markov inequality where A_n is precisely determined. In these results A_n is expressed in terms of the *weighted* Chebyshev polynomials. The first of these cases, studied by Frappier [6], involved taking the supremum over polynomials with restrictions on the location of their zeros with $w \equiv 1$.

The second case, when $w(x) = e^{-x^2}$, $\|\cdot\|$ is the sup-norm on the whole real line $(-\infty, \infty)$, and the supremum is taken over $\mathscr{P}^r_n$, was first considered by Mohapatra *et al.* [14] and later by Li *et al.* [10]. In [10] the A_n is determined in terms of $T_n(\cdot, e^{-x^2})$, the weighted Chebyshev polynomial of degree n associated with the Hermite weight e^{-x^2} on $(-\infty, \infty)$. Precisely, the following result is proved:

THEOREM A. *Let $T_n = T_n(\cdot, e^{-x^2})$ and $\|\cdot\| = \|\cdot\|_{(-\infty,\infty)}$. Then*

$$\max_{p \in \mathscr{P}^r_n, p \neq 0} \frac{\|(e^{-x^2}p)'\|}{\|e^{-x^2}p\|} = \frac{\|(e^{-x^2}T_n)'\|}{\|e^{-x^2}T_n\|}.$$

Many results on estimates of A_n associated with Markov-type inequalities have been obtained (cf. [12, pp. 700–723] and the references therein).

One of the most recent of such results was a rational version of the Markov-type inequality with the sup-norm, namely when $\|\cdot\| = \|\cdot\|_{[-1,1]}$ and the weight $w \equiv 1/q$, where q is a polynomial of degree n with distinct zeros outside of $[-1, 1]$. This case was first studied by Borwein *et al.* [4] and later by Min [13]. Although A_n is still unknown, estimates have been found in terms of the nth Chebyshev rational function that can be viewed as a weighted Chebyshev polynomial.

In this paper we will concentrate on the Laguerre weight e^{-x} with $\|\cdot\| = \|\cdot\|_{[0,\infty)}$ and obtain A_n in terms of the weighted Chebyshev polynomials. We will adapt the approach of Markov (see, for example [1, 10, 19]) by considering first a pointwise version of the problem and viewing this new problem as that of minimizing some linear functional under appropriate constraints. Before we state our result we mention some related results for Markov-type inequalities involving Laguerre weight.

In 1964, Szegő [17] proved the following:

THEOREM B. *For $p \in \mathscr{P}_n^r$,*

$$\|p'(x)\, e^{-x}\|_{[0,\infty)} \leqslant M_n\, \|p(x)\, e^{-x}\|_{[0,\infty)},$$

where $M_n \sim n$; i.e., M_n/n is bounded away from zero and infinity.

This result of Szegő implies an estimate of A_n, viz. $A_n = O(n)$. For L^2 norm, the corresponding sharp Markov-type inequality associated with the Laguerre weight is obtained by Turan [18]. This result was rediscovered by Shampine [15] who also obtained best constants for similar inequalities where second derivatives are used. Further results and references can be found in Dörfier's paper [5].

Markov-type inequalities in L^p spaces associated with weights $e^{-|x|^\alpha}$, $\alpha > 1$, have been considered in [7–9], but only estimates of A_n are obtained. We remark that even in the L^2 Markov inequality on $[-1, 1]$,

$$\left(\int_{-1}^{1} |p'(x)|^2\, dx\right)^{1/2} \leqslant A_n \left(\int_{-1}^{1} |p(x)|^2\, dx\right)^{1/2}, \qquad p \in \mathscr{P}_n,$$

the extremal polynomials that reduce the inequality to an equality have not yet been found (see [2, 7]).

We now turn to our main result.

2. THE MAIN RESULT

Let w be a weight function. The weighted Chebyshev polynomial $T_n(x; w) = x^n + \textit{lower degree terms} \in \mathscr{P}_n^r$ can be characterized by $w(x)\, T_n(x; w)$ having an alternating set of $n+1$ points; see for example [11]. We will use $\hat{T}_n$ to denote the normalized weighted Chebyshev polynomial; that is, $\hat{T}_n = T_n/\|wT_n\|$.

The main result of this paper is the following:

THEOREM 1. *With* $w(x)=e^{-x}$, $p_n \in \mathscr{P}_n^r$, *and* $\|\cdot\| = \|\cdot\|_{[0,\infty)}$,

$$A_n = \sup\{\|(wp_n)'\| : \|wp_n\| = 1,\ p_n \in \mathscr{P}_n^r\} = \|(w\hat{T}_n)'\| = |(w\hat{T}_n)'(0)|.$$

Remark. Notice that the norm $\|\cdot\|$ is always taken on weighted polynomials wp_n and $(wp_n)' = w(p_n' - p_n)$. By a well-known result of Mhaskar and Saff (cf. [11]), for $w(x)=e^{-x}$, we have

$$\|wp_n\|_{[0,\infty)} = \|wp_n\|_{[0,\pi n/2]} \qquad \text{and} \qquad \|(wp_n)'\|_{[0,\infty)} = \|(wp_n)'\|_{[0,\pi n/2]}.$$

Thus, we can use the compact intervals $[0, \pi n/2]$ to replace $[0,\infty)$ when dealing with the sup-norm.

The outline on the proof goes like this: We will first consider the following pointwise extremal problem: For $y \geqslant 0$,

$$(P_y) \qquad \begin{cases} \text{Maximize } (wp_n)'(y) \\ \text{subject to} \\ p_n \in \mathscr{P}_n^r \text{ and } \|wp_n\| \leqslant 1. \end{cases}$$

A standard compact argument shows that the extremal solutions of (P_y) always exists. Let $M_n(y)$ denote the maximum value obtained while solving the problem (P_y). Then to find A_n, we find the largest value of $M_n(y)$ as a function of y. It turns out that extremal solutions of (P_y) for the maximum points y of $M_n(y)$ are the weighted Chebyshev polynomials.

To prove Theorem 1, we need the following result of Shapiro [16]. Let P be a normed space with norm $\|\cdot\|$ and let L be a linear functional on P. Consider the more general problem

$$(P_L) \qquad \begin{cases} \text{Maximize } L(p) \\ \text{subject to} \\ p \in P \text{ and } \|p\| \leqslant 1. \end{cases} \tag{3}$$

LEMMA 2 (Representation Theorem). *Let $C(S)$ be the set of real-valued continuous functions on a compact Hausdorff space S. Let P be an n-dimensional linear subspace of $C(S)$ over $\mathbb{R}$. Let $\|\cdot\|$ be the sup-norm taken on the set S. Let $L \neq 0$ be a real-valued linear functional on P. Then there* *exist points* $\tau_1, \tau_2, \ldots, \tau_r \in S$ *and nonzero real numbers* $\lambda_1, \lambda_2, \ldots, \lambda_r$, *where* $1 \leqslant r \leqslant n$ *such that*

$$L(p) = \sum_{i=1}^{r} \lambda_i p(\tau_i), \qquad p \in P,$$

and

$$\|L\| = \sum_{i=1}^{r} |\lambda_i|,$$

where

$$\|L\| := \sup\{|L(p)| \colon \|p\| \leqslant 1, p \in P\}.$$

The proof of this result can be found in [3, 16].

Now, let us go back to the pointwise problem (P_y) for $w(x) = e^{-x}$. Letting $P = w\mathscr{P}_n^r := \{wp_n \colon p_n \in \mathscr{P}_n^r\}$ and $L(wp) = (wp)'(y)$ and taking $S = [0, n\pi/2]$ (in view of the Remark following Theorem 1) in the Representation Theorem, we have the following:

LEMMA 3. *The weighted polynomial* $wQ_n \in w\mathscr{P}_n^r$ *is an extremal element for* (P_y) *if and only if there exist* $\lambda_j \neq 0$ *and* τ_j, $j = 1, 2, \ldots, r$ *for some* $r = r(y)$, $1 \leqslant r \leqslant n+1$, *with* $0 \leqslant \tau_1 < \tau_2 < \cdots < \tau_r \leqslant n\pi/2$ *such that*

$$\begin{aligned} &(wp_n)'(y) = \sum_{i=1}^{r} \lambda_j (wp_n)(\tau_j) \qquad \textit{for all} \quad p_n \in \mathscr{P}_n^r, \\ &\operatorname{sgn} \lambda_j = \operatorname{sgn}(wQ_n)(\tau_j), \qquad \textit{and} \\ &|(wQ_n)(\tau_j)| = \|wQ_n\| = 1, \qquad j = 1, 2, \ldots, r. \end{aligned} \tag{4}$$

Remarks. (i) This result completely characterizes the extremal elements for (P_y). (ii) The last condition in (4) suggests that Chebyshev polynomials will be of interest.

3. ADDITIONAL LEMMAS

Let $w(x) = e^{-x}$. In this section, we shall obtain lemmas which will be used in the proof of our theorem.

Now let $y \geqslant 0$ and let $w(x)\, Q_n(x) := w(x)\, Q_n(x, y)$ be an extremal element for (P_y).

Our first lemma says something about the case where $r = n+1$ in Lemma 3.

LEMMA 4. *If $r = n+1$, then $Q_n = \hat{T}_n$ or $-\hat{T}_n$.*

Proof. First note that since $(wQ_n)' = w(Q_n' - Q_n)$, $(wQ_n)'$ has at most n zeros. Now suppose there is at least one j such that $wQ_n(\tau_j) = wQ_n(\tau_{j+1})$. If this is the case, then $(wQ_n)'(x)'$ has a zero in (τ_j, τ_{j+1}). But $(wQ_n)'(\tau_j) = 0$ for $j = 2, 3, ..., n+1$. So $(wQ_n)'$ has at least $n+1$ zeros, a contradiction. Hence, Q_n must have the maximal equioscillation property enjoyed by only $\pm\hat{T}_n$. ▮

The next result tells us more about the zeros of $w(x)\,p(x)$ and its derivative when $p \in \mathscr{P}_n^r$.

LEMMA 5. *Let $p_n \in \mathscr{P}_n^r$ have n simple zeros in $[0, \infty)$. Then $(wp_n)'$ has exactly n zeros in $(0, \infty)$. Further, the zeros of $(wp_n)'$ in $(0, \infty)$ and the zeros of p_n interlace so that the largest zero of p_n is less than the largest zero of $(wp)'$.*

Proof. First note that $(wp_n)' = w(p_n' - p_n)$. Let $q = p_n' - p_n$. It is obvious that q can have at most n zeros. We now show that q has n distinct zeros in $(0, \infty)$.

Let x_1 and x_2 be two consecutive zeros of p_n. Then $q(x_1) = p_n'(x_1)$ and $q(x_2) = p_n'(x_2)$ have opposite signs as implied by the assumption that p_n has n simple zeros. Thus, q and therefore $(wp_n)'$ has a zero between every two consecutive zeros of p. So, if x^* denotes the largest zero of p_n, then q has at least $n-1$ distinct zeros in $(0, x^*)$. Suppose, without loss of generality, that the leading coefficient of p_n is positive. Then, since the degree of p_n is one more than the degree of p_n' we see that $q(x) < 0$ for sufficiently large x. But $q(x^*) = p_n'(x^*) > 0$. So q has one more zero in (x^*, ∞). Thus, we have accounted for n distinct zeros of q in $(0, \infty)$. These zeros interlace with those of p_n, with the largest zero of q being larger than the largest zero of p_n. ▮

LEMMA 6. *For problem (P_0), that is, when $y = 0$ in the problem (P_y), we have $r = n+1$ in Lemma 3, and hence by Lemma 4, we have $Q_n(x, 0) = \pm\hat{T}_n$.*

Proof. Suppose $y = 0$ and $r \leqslant n$. Choose $\tau_{r+1}, \tau_{r+2}, ..., \tau_n$ each in $(0, \infty)$ so that $\tau_i \neq \tau_j$, $i \neq j$. Let $p_n(x) := \prod_{i=1}^n (x - \tau_i)$. By Lemma 5 we have that $(wp_n)'(x)$ has exactly n simple zeros, all of which are in (τ_1, ∞). But by (4), we have that y is a zero of $(wp_n)'$. Since $y = 0$, $y \notin (\tau_1, \infty)$. So $(wp_n)'$ has $n+1$ distinct zeros, a contradiction. ▮

4. PROOF OF THE MAIN RESULT

Recall that $w(x)=e^{-x}$ and $\|\cdot\|=\|\cdot\|_{[0,\infty)}$.
First note that

$$\begin{aligned}&\sup\{\|(wp)'\|_{[0,\infty)}: \|wp\|_{[0,\infty)}\leqslant 1,\ p\in\mathscr{P}_n^r\}\\&\quad=\sup\{\|(wp)'\|_{[a,\infty)}: \|wp\|_{[a,\infty)}\leqslant 1,\ p\in\mathscr{P}_n^r\},\end{aligned}$$

for any $a\in\mathbb{R}$. This follows from the fact that $w(x-a)\,p(x-a)=w(x)\times e^a p(x-a)\in w\mathscr{P}_n$. This translation invariance property of the problem will be crucial in our proof. We explore this property as follows.

Let $y>0$. Let $Q_n(x,y)$ be an extremal polynomial in x for the pointwise problem (P_y). Consider $p_n(x):=w(y)\,Q_n(x+y,y)$. Note that

$$\|w(x)\,p_n(x)\|=\|w(x+y)\,Q_n(x+y,y)\|\leqslant 1.$$

Further, note that by Lemma 6

$$\begin{aligned}\frac{\partial}{\partial x}w(x)\,Q_n(x,y)\bigg|_{x=y}&=\frac{\partial}{\partial x}w(x)\,p_n(x)\bigg|_{x=0}\\&\leqslant\frac{\partial}{\partial x}w(x)\,Q_n(x,0)\bigg|_{x=0}\\&=|(w\hat{T}_n)'\,(0)|.\end{aligned}$$

Thus, we see that

$$\begin{aligned}&\sup\{\|(wp_n)'\|: \|wp_n\|\leqslant 1,\ p_n\in\mathscr{P}_n^r\}\\&\quad=\max\left\{\frac{\partial}{\partial x}w(x)\,Q_n(x,y)\bigg|_{x=y}: y\geqslant 0\right\}=|(w\hat{T}_n)'\,(0)|.\end{aligned}$$

On the other hand, note that for some $y_0\geqslant 0$,

$$|(w\hat{T}_n)'\,(0)|\leqslant\|(w\hat{T}_n)'\|=|(w\hat{T}_n)'\,(y_0)|\leqslant\left|\frac{\partial}{\partial x}w(x)_n(x,y_0)\bigg|_{x=y_0}\right|\leqslant|(w\hat{T}_n)'\,(0)|.$$

So we have

$$|(w\hat{T}_n)'\,(0)|=\|(w\hat{T}_n)'\|.$$

This completes our proof.

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